

# Codimension one foliations with Bott-Morse singularities I

Bruno Scárdua and José Seade

## Abstract

We study codimension one (transversally oriented) foliations  $\mathcal{F}$  on oriented closed manifolds  $M$  having non-empty compact singular set  $\text{sing}(\mathcal{F})$  which is locally defined by Bott-Morse functions. We prove that if the transverse type of  $\mathcal{F}$  at each singular point is a center and  $\mathcal{F}$  has a compact leaf with finite fundamental group or a component of  $\text{sing}(\mathcal{F})$  has codimension  $\geq 3$  and finite fundamental group, then all leaves of  $\mathcal{F}$  are compact and diffeomorphic,  $\text{sing}(\mathcal{F})$  consists of two connected components, and there is a Bott-Morse function  $f : M \rightarrow [0, 1]$  such that  $f : M \setminus \text{sing}(\mathcal{F}) \rightarrow (0, 1)$  is a fiber bundle defining  $\mathcal{F}$  and  $\text{sing}(\mathcal{F}) = f^{-1}(\{0, 1\})$ . This yields to a topological description of the type of leaves that appear in these foliations, and also the type of manifolds admitting such foliations. These results unify, and generalize, well known results for cohomogeneity one isometric actions and a theorem of Reeb for foliations with Morse singularities of center type. In this case each leaf of  $\mathcal{F}$  is a sphere fiber bundle over each component of  $\text{sing}(\mathcal{F})$ .

## Introduction

Cohomogeneity one isometric actions of Lie groups play an important role in Differential Geometry, particularly in the Theory of Minimal Submanifolds (see for instance [9]). A basic well-known fact about these actions is that whenever the group and the manifold are compact, if all orbits are principal then the space of orbits is  $S^1$ , and if there are special orbits then there are exactly two of them and the space of orbits is the interval  $[0, 1]$ . Notice that such an action defines a codimension one foliation with compact leaves and singular set the special orbits. Since the action is isometric, the intersection of the orbits with a slice  $\Sigma$  transverse to a special orbit corresponds to a Morse singularity of center type.

From the Foliation Theory viewpoint this reminds us of two important results of Reeb. The first of them is the Complete Stability Theorem, which states that a transversely oriented non-singular codimension one foliation having a compact leaf with finite fundamental group on a closed manifold, is a fibration over the circle. The second result concerns foliations with non-empty singular set. It states that if a codimension one transversely oriented foliation on a closed manifold has only Morse (isolated) singularities of center type then there are exactly two such singularities and the manifold is homeomorphic to a sphere.

In this article we unify these two situations by introducing the concept of foliation with Bott-Morse singularities. This means that the singular set of such a foliation is a disjoint union of compact submanifolds and in a neighborhood of each singular point the foliation is defined by a Bott-Morse function; so it is a usual Morse function restricted to each transversal slice. Given such a foliation, the transverse type of each connected component of the singular set  $\text{sing}(\mathcal{F})$  is well-defined, and we can speak of components of center type, of saddle type, etc., according to the Morse index of the foliation on a transversal slice.

We prove the following Complete Stability Theorem:

**Theorem A.** *Let  $\mathcal{F}$  be a smooth foliation with Bott-Morse singularities on a closed oriented manifold  $M$  of dimension  $m \geq 3$  having only center type components in  $\text{sing}(\mathcal{F})$ . Assume that  $\mathcal{F}$  has some compact leaf  $L_o$  with finite fundamental group, or there is a codimension  $\geq 3$  component  $N$  with finite fundamental group. Then all leaves of  $\mathcal{F}$  are compact, stable, with finite fundamental group. If, moreover,  $\mathcal{F}$  is transversally orientable, then  $\text{sing}(\mathcal{F})$  has exactly two components and there is a differentiable Bott-Morse function  $f: M \rightarrow [0, 1]$  whose critical values are  $\{0, 1\}$  and such that  $f|_{M \setminus \text{sing}(\mathcal{F})}: M \setminus \text{sing}(\mathcal{F}) \rightarrow (0, 1)$  is a fiber bundle with fibers the leaves of  $\mathcal{F}$ .*

The proof of Theorem A actually shows that every compact transversely oriented foliation with non-empty singular set, all of Bott-Morse type, has exactly two components in its singular set and is given by a Bott-Morse function  $f: M \rightarrow [0, 1]$  as is in the statement.

The first step for proving Theorem A is the following Local Stability Theorem:

**Theorem B.** *Let  $\mathcal{F}$  be a smooth codimension one foliation on a closed, oriented manifold  $M^m$  having Bott-Morse singularities and let  $N^n \subset$*

$\text{sing}(\mathcal{F})$  be a (compact) component with finite holonomy group (e.g., if  $N$  has finite fundamental group). Then there exists a neighborhood  $W$  of  $N$  in  $M$  where  $\mathcal{F}$  is given by a Bott-Morse function  $f: W \rightarrow \mathbb{R}$ . If moreover the transverse type of  $\mathcal{F}$  along  $N$  is a center, then  $N$  is stable and the leaves of  $\mathcal{F}$  in  $W$ , for a suitable choice of  $W$ , are fiber bundles over  $N$  with fiber  $S^{m-n-1}$ .

Theorem A and its proof lead to the following generalization of Theorem 1.5 in [11]:

**Theorem C.** *Let  $\mathcal{F}$  be a transversally oriented, compact foliation with Bott-Morse singularities on a closed, oriented, connected manifold  $M^m$ ,  $m \geq 3$ , with non-empty singular set  $\text{sing}(\mathcal{F})$ . Let  $L$  be a leaf of  $\mathcal{F}$ . Then  $\text{sing}(\mathcal{F})$  has two connected components  $N_1, N_2$ , both of center type, and one has:*

- (i)  $M \setminus (N_1 \cup N_2)$  is diffeomorphic to the cylinder  $L \times (0, 1)$ .
- (ii)  $L$  is a sphere fiber bundle over both manifolds  $N_1, N_2$  and  $M$  is diffeomorphic to the union of the corresponding disc bundles over  $N_1, N_2$ , glued together along their common boundary  $L$  by some diffeomorphism  $L \rightarrow L$ .
- (iii) In fact one has a double-fibration

$$N_1 \xleftarrow{\pi_1} L \xrightarrow{\pi_2} N_2 ,$$

and  $M$  is homeomorphic to the corresponding mapping cylinder, i.e., to the quotient space of  $(L \times [0, 1]) \cup (N_1 \cup N_2)$  by the identifications  $(x, 0) \sim \pi_1(x)$  and  $(x, 1) \sim \pi_2(x)$ .

This yields to a description of this type of foliations on manifolds of dimensions 3 and 4 (see Section 4). In dimension 3 our results imply that every foliation as in Theorem C is actually given by a cohomogeneity one action of either  $SO(3)$  or the 2-torus  $S^1 \times S^1$ .

Unless it is stated otherwise, in this work all manifolds, bundles, foliations and maps are assumed to be of class  $C^\infty$ . This is just for simplicity, because essentially everything we say holds in class  $C^r$ , for all  $r \geq 1$ .

In Section 1 we give the precise definition of foliation with Bott-Morse singularities and discuss key-examples of such foliations.

This work was done during a visit of the first named author to the Instituto de Matemáticas of UNAM in Cuernavaca, Mexico, and a visit

of the second named author to IMPA, Rio de Janeiro, Brazil. The authors want to thank these institutions for their support and hospitality.

## 1 Definitions and examples

Let  $\mathcal{F}$  be a codimension one smooth foliation on a manifold  $M$  of dimension  $m \geq 2$ . We denote by  $\text{sing}(\mathcal{F})$  the singular set of  $\mathcal{F}$ . We say that the singularities of  $\mathcal{F}$  are of *Bott-Morse type* if  $\text{sing}(\mathcal{F})$  is a disjoint union of a finite number of disjoint compact connected submanifolds,  $\text{sing}(\mathcal{F}) = \bigcup_{j=1}^t N_j$ , each of codimension  $\geq 2$ , and for each

$p \in N_j \subset \text{sing}(\mathcal{F})$  there exists a neighborhood  $V$  of  $p$  in  $M$  and a diffeomorphism  $\varphi: V \rightarrow P \times D$ , where  $P \subset \mathbb{R}^n$  and  $D \subset \mathbb{R}^{m-n}$  are discs centered at the origin, such that  $\varphi$  takes  $\mathcal{F}|_V$  into the product foliation  $P \times \mathcal{G}$ , where  $\mathcal{G} = \mathcal{G}(N_j)$  is the foliation on  $D$  given by some Morse function singularity at the origin. In other words,  $\text{sing}(\mathcal{F}) \cap V = N_j \cap V$ ,  $\varphi(N_j \cap V) = P \times \{0\} \subset P \times D \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$  and we can find coordinates  $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_{m-n}) \in V$  such that  $N_j \cap V = \{y_1 = \dots = y_{m-n} = 0\}$  and  $\mathcal{F}|_V$  is given by the levels of a function  $J_{N_j}(x, y) = \sum_{j=1}^{m-n} \lambda_j y_j^2$  where  $\lambda_j \in \{\pm 1\}$ .

The discs  $\Sigma_p = \varphi^{-1}(x(p) \times D)$  are transverse to  $\mathcal{F}$  outside  $\text{sing}(\mathcal{F})$  and the restriction  $\mathcal{F}|_{\Sigma_p}$  is an ordinary Morse singularity, whose Morse index does not depend on the point  $p$  in the component  $N_j$ . We shall refer to  $\mathcal{G}(N_j) = \mathcal{F}|_{\Sigma_p}$  as the *transverse type* of  $\mathcal{F}$  along  $N_j$ . This is a codimension one foliation in the disc  $\Sigma_p$  with an ordinary Morse singularity at  $\{p\} = N_j \cap \Sigma_p$ .

If  $N_j$  has dimension zero (or if we look at a transversal slice), then  $\mathcal{F}$  has an ordinary Morse singularity at  $p$  and for suitable local coordinates,  $\mathcal{F}$  is given by the level sets of a quadratic form  $f = f(p) - (y_1^2 + \dots + y_r^2) + y_{r+1}^2 + \dots + y_m^2$ , where  $r \in \{0, \dots, m\}$  is the *Morse index* of  $f$  at  $p$ . The Morse singularity  $p$  is a *center* if  $r$  is 0 or  $m$ , otherwise  $p$  is called a *saddle*. In a neighborhood of a center, the leaves of  $\mathcal{F}$  are diffeomorphic to  $(m-1)$ -spheres. In a neighborhood of a saddle  $q$ , we have conical leaves called *separatrices* of  $\mathcal{F}$  through  $q$ , which are given by expressions  $y_1^2 + \dots + y_r^2 = y_{r+1}^2 + \dots + y_m^2 \neq 0$ . Each such leaf contains  $p$  in its closure.

**Definition 1.1.** A component  $N \subset \text{sing}(\mathcal{F})$  is of *center type* (or just a *center*) if the transverse type  $\mathcal{G}(N) = \mathcal{F}|_{\Sigma_q}$  of  $\mathcal{F}$  along  $N$  is a center.

Similarly, the component  $N \subset \text{sing}(\mathcal{F})$  is of *saddle type* if its transverse type is a saddle.

As in the case of isolated singularities, these concepts do not depend on the choice of orientations. We denote by  $C(\mathcal{F}) \subset \text{sing}(\mathcal{F})$  the union of center type components in  $\text{sing}(\mathcal{F})$ , and by  $S(\mathcal{F})$  the corresponding union of saddle components. Of course saddles can have different transversal Morse indices; this will be relevant for Part II of this article [19].

**Definition 1.2.** We say that  $\mathcal{F}$  is *compact* if every leaf of  $\mathcal{F}$  is compact (and consequently  $S(\mathcal{F}) = \emptyset$ ). The foliation  $\mathcal{F}$  is *proper* if every leaf of  $\mathcal{F}$  is closed off  $\text{sing}(\mathcal{F})$ .

If  $\mathcal{F}$  is proper and  $M$  is compact, then all leaves are compact except for those containing separatrices of saddles in  $S(\mathcal{F})$  and such a leaf is contained in a compact singular variety  $\overline{L} = \overline{L} \cap \text{sing}(\mathcal{F}) \subset L \cup S(\mathcal{F})$ . A proper foliation on a compact manifold is compact if and only if  $S(\mathcal{F}) = \emptyset$ .

Let  $N \subset C(\mathcal{F})$  be a component of dimension  $k$ . Suppose that the nearby leaves of  $\mathcal{F}$  are compact. We define  $\Omega(N, \mathcal{F}) = \Omega(N) \subset M$  as the union of  $N$  and all the leaves  $L \in \mathcal{F}$  which are compact and bound a compact invariant region  $R(L, N)$  which is a neighborhood of  $N$  in  $M$ . The region  $R(L, N)$  is equivalent to a fibre bundle with fibre the closed disc  $\overline{D}^{m-k}$  over  $N$ , the fibers being transversal to the leaves of  $\mathcal{F}$ . As we will see, the notion of holonomy of the singular set, to be introduced in section 2.1, assures that if  $N$  is of center type and has finite holonomy group (e.g., if  $\pi_1(N)$  is finite) then  $\Omega(N, \mathcal{F})$  is an open subset of  $M$ .

**Definition 1.3 (orientability and transverse orientability).** Let  $\mathcal{F}$  be a codimension one foliation with Bott-Morse singularities on  $M^m$ ,  $m \geq 2$ . The foliation  $\mathcal{F}$  is *orientable* if there exists a one-form  $\Omega$  on  $M$  such that  $\text{sing}(\mathcal{F}) = \text{sing}(\Omega)$ , and  $\mathcal{F}$  coincides with the foliation defined by  $\Omega = 0$  outside the singular set. The choice of such a one-form  $\Omega$  is called an *orientation* for  $\mathcal{F}$ . We shall say that  $\mathcal{F}$  is *transversally orientable* if there exists a vector field  $X$  on  $M$ , possibly with singularities at  $\text{sing}(\mathcal{F})$ , such that  $X$  is transverse to  $\mathcal{F}$  outside  $\text{sing}(\mathcal{F})$ .

The following basic result is easily proved using the fact that we can always choose local orientations for  $\mathcal{F}$ , and also orientations along paths which are null-homotopic.

**Proposition 1.4.** *Let  $\mathcal{F}$  be a codimension one foliation with Bott-Morse singularities on  $M^m$ ,  $m \geq 2$ . Suppose  $M$  is orientable. Then:*

- (i) *The foliation  $\mathcal{F}$  is orientable if and only if it is transversally orientable.*
- (ii) *If  $M$  is simply-connected, then  $\mathcal{F}$  is transversally orientable.*

## 1.1 Examples

Basic examples of foliations with Bott-Morse singularities are given by Bott-Morse functions and by products of Morse foliations by closed manifolds. Next we give four types of examples of foliations with Bott-Morse singularities.

**Example 1 (Fiber bundles).** Let  $\widetilde{M}^{m+k}$  and  $M^m$  be connected oriented manifolds. Let  $\mathcal{F}$  be a foliation with Bott-Morse singularities on  $M$  and let  $\pi: \widetilde{M} \rightarrow M$  be a proper submersion. Then the pull-back foliation  $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$  has only Bott-Morse singularities; hence  $\widetilde{\mathcal{F}}$  is a foliation with Bott-Morse singularities and its transverse type at each component is that of  $\mathcal{F}$  at the corresponding point.

For instance, take a vector field on an oriented closed surface  $S$  with non-degenerate singularities, and consider the corresponding foliation  $\mathcal{L}$ . Given any  $S^1$ -bundle  $\pi: M \rightarrow S$ , the pull-back foliation  $\mathcal{F} = \pi^*(\mathcal{L})$  has Bott-Morse singularities on  $M$ ;  $\text{sing}(\mathcal{F})$  is a union of circles.

In particular, the Hopf fibration  $\pi: S^3 \rightarrow S^2$  gives rise, in this way, to Bott-Morse foliations on  $S^3$ . We can consider also  $SO(3)$ , regarded as the unit tangent bundle of  $S^2$ , to get examples on  $SO(3) \cong \mathbb{R}P^3$ .

**Example 2 (Mapping cylinders and Lens spaces).** Consider now a closed oriented manifold  $L$  that fibers as a sphere fiber bundle over two other manifolds  $N_1$  and  $N_2$ , of possibly different dimensions. Let  $E_1, E_2$  be the corresponding disc bundles. Then each  $E_i$  can be foliated by copies of  $L$  by taking concentric spheres in the corresponding fibers. We may now glue  $E_1$  and  $E_2$  by some diffeomorphism of the common boundary  $L$  to get a closed oriented manifold  $M$  with a foliation with Bott-Morse singularities at  $N_1$  and  $N_2$ , both of center type.

For instance, take two solid torii  $S^1 \times D^2$ , equipped with the same foliation, and glue their boundaries by a diffeomorphism that carries a meridian of the first torus into a curve on the second which is homologous to  $q$ -meridians and  $p$ -longitudes, with  $p, q \geq 1$  coprime. We

obtain foliations with Bott-Morse singularities on the so-called *Lens spaces*  $L(p, q)$ .

**Example 3 (Cohomogeneity one actions).** As mentioned before, a cohomogeneity one isometric action leads naturally to compact foliations with Bott-Morse singularities of center type.

For instance [11], consider  $SO(n+1, \mathbb{R})$  as a subgroup of  $SO(n+1, \mathbb{C})$ . The standard action of this group on  $\mathbb{C}^{n+1}$  defines an action of  $SO(n+1, \mathbb{R})$  on  $\mathbb{C}P(n)$ , which is by isometries with respect to the Fubini-Study metric.

The special orbits are the complex quadric  $Q_{n-1} \subset \mathbb{C}P(n)$ , of points with homogeneous coordinates satisfying  $\sum_{j=0}^n z_j^2 = 0$ , and the real projective space  $\mathbb{R}P(n) \subset \mathbb{C}P(n)$ , consisting of the points which are fixed by the involution in  $\mathbb{C}P(n)$  given by complex conjugation.

The principal orbits are copies of the flag manifold

$$F_+^{n+1}(2, 1) \cong SO(n+1, \mathbb{R}) / (SO(n-1, \mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})),$$

of oriented 2-planes in  $\mathbb{R}^{n+1}$  and (unoriented) lines in these planes. Each such orbit splits  $\mathbb{C}P(n)$  in two pieces, each being a tubular neighborhood of a special orbit.

The case  $n = 2$  is specially interesting because this provides an equivariant version of the Arnold-Kuiper-Massey theorem that  $\mathbb{C}P(2)$  modulo conjugation is the 4-sphere, see for instance [11]. This is also proved in [2] and [1], where there are also interesting generalizations of these constructions and theorem to the quaternionic and the octonian projective planes.

**Example 4 (Poisson manifolds).** A *Poisson structure* on a smooth manifold  $M$  consists of a Lie algebra structure on the ring of functions  $C^\infty(M)$ , generalizing the classical Poisson bracket on a symplectic manifold, which satisfies a Leibniz identity in such a way that  $\{, h\}$  is a derivation. There is thus a vector bundle morphism  $\psi: T^*M \rightarrow TM$  associated with  $\{, \}$ , satisfying an integrability condition, whose rank at each point is called the rank of the Poisson structure.

If the rank is constant, then the integrability condition implies one has a foliation on  $M$ , of dimension equal to the rank, and the tangent space of the foliation is, at each point  $x \in M$ , the image of  $\psi(T_x^*M)$  in  $T_xM$ . If the rank is not constant, then one still has a generalized foliation in the sense of [20], *i.e.*, a foliation with singularities at the

points where the rank drops, but at each such point one has a leaf of dimension the corresponding rank, whose tangent space is again given by  $\psi(T_x^*M)$ . The Dolbeault-Weinstein theorem implies that at such points the transversal structure plays a key role (see [21]).

It would be interesting to study Poisson structures for which the corresponding foliation has Bott-Morse singularities (*cf.* [6] for instance.)

## 2 Holonomy and local stability

The notion of stability plays a fundamental role in the classical theory of (nonsingular) foliations. In what follows we bring this notion into our framework.

**Definition 2.1.** Let  $\mathcal{F}$  be a (possibly singular) foliation on  $M$ . A subset  $B \subset M$ , invariant by  $\mathcal{F}$ , is *stable* (for  $\mathcal{F}$ ) if for any given neighborhood  $W$  of  $B$  in  $M$  there exists a neighborhood  $W' \subset W$  of  $B$  in  $M$  such that every leaf of  $\mathcal{F}$  intersecting  $W'$  is contained in  $W$ .

The following technical result comes from the proof of the Complete Stability theorem of Reeb (*cf.* [7]):

**Lemma 2.2.** *Let  $\mathcal{F}$  be a codimension one (nonsingular) foliation on  $M$ .*

- (i) *Let  $L$  be a compact leaf of  $\mathcal{F}$  and let  $L_n$  be a sequence of compact leaves of  $\mathcal{F}$  accumulating to  $L$ . Then given a neighborhood  $W$  of  $L$  in  $M$  one has  $L_n \subset W$  for all  $n$  sufficiently large.*
- (ii) *Denote by  $L_x$  the leaf of  $\mathcal{F}$  containing  $x \in M$  and define  $\widehat{M}$  as the subset of points  $x \in M$  such that the leaf  $L_x$  is compact with finite fundamental group.*

*Then any leaf contained in  $\partial\widehat{M}$  is closed in  $M$ .*

According to [7], Proposition 2.20, page 103, in case  $\mathcal{F}$  is a compact foliation without singularities, stability of a (compact) leaf is equivalent to finiteness of its holonomy group. We will extend this result for compact codimension one foliations with Bott-Morse singularities (see Proposition 2.7) using the following notion.

### 2.1 Holonomy of the singular set

Given a component  $N \subset \text{sing}(\mathcal{F})$  we consider a collection  $\mathcal{U} = \{U_j\}_{j \in J}$  of open subsets  $U_j \subset M$  and charts  $\varphi_j: U_j \rightarrow \varphi_j(U_j) \subset \mathbb{R}^m$  with the following properties:



- (1) Each  $\varphi_j: U_j \rightarrow \varphi_j(U_j) \subset \mathbb{R}^m$  defines a local product trivialization of  $\mathcal{F}$ ,  $U_j \cap N$  is a disc and  $\varphi_j(U_j)$  is a product of discs.
- (2)  $\bigcup_{j \in J} U_j$  is an open neighborhood of  $N$  in  $M$ .
- (3) If  $U_i \cap U_j \neq \emptyset$  then there exists an open subset  $U_{ij} \subset M$  containing  $U_i \cup U_j$  and a chart  $\varphi_{ij}: U_{ij} \rightarrow \varphi_{ij}(U_{ij}) \subset \mathbb{R}^m$  of  $M$ , such that  $\varphi_{ij}$  defines a product structure for  $\mathcal{F}$  in  $U_{ij}$  and  $U_{ij} \cap N \supset (U_i \cup U_j \cap N) \neq \emptyset$ .

Such a covering  $\mathcal{U}$  will be called a *chain adapted* to  $\mathcal{F}$  and  $N$ . When  $N$  is compact we can assume  $\mathcal{U}$  to be finite say  $\mathcal{U} = \{U_1, \dots, U_{\ell+1}\}$ . Suppose now that  $U_j \cap U_{j+1} \neq \emptyset$ ,  $\forall j \in \{1, \dots, \ell\}$ . In each  $U_j$  we choose a transverse disc  $\Sigma_j$ ,  $\Sigma_j \cap N = \{q_j\}$  such that  $\Sigma_{j+1} \subset U_j \cap U_{j+1}$  if  $j \in \{1, \dots, \ell\}$ . By choice of  $\mathcal{U}$  in each  $U_j$  the foliation is given by a smooth function  $F_j: U_j \rightarrow \mathbb{R}$  which is the natural trivial extension of its restriction to any of the transverse discs  $\Sigma_j$  or  $\Sigma_{j+1}$ .

There is a  $C^\infty$  local diffeomorphism  $\varphi_j: (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$  such that  $F_{j+1}|_{\Sigma_{j+1}} = \varphi_j \circ F_j|_{\Sigma_{j+1}}$ . This implies that  $F_{j+1} = \varphi_j \circ F_j$  in  $U_j \cap U_{j+1}$  (notice that by condition (3) if  $U_i \cap U_k \neq \emptyset$  then every plaque  $\mathcal{F}$  in  $U_i \setminus N$  intersects at most one plaque of  $U_k \setminus N$ ).

**Definition 2.3.** The *holonomy map associated to the chain*  $\mathcal{U} = \{U_1, \dots, U_\ell\}$  is the local diffeomorphism  $\varphi: (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$  defined by the composition  $\varphi = \varphi_\ell \circ \dots \circ \varphi_1$ .

Given now a path  $c: [0, 1] \xrightarrow{C^0} N$  we can find a finite chain  $\mathcal{U} = \{U_1, \dots, U_{\ell+1}\}$  such that  $\bigcup_{j=1}^{\ell+1} U_j \supset c([0, 1])$  and define the *holonomy map* of  $c: [0, 1] \rightarrow N$  as  $\varphi = \varphi_\ell \circ \dots \circ \varphi_1: (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ . Clearly if  $\tilde{c}: [0, 1] \rightarrow N$  is  $C^0$ -close to  $c: [0, 1] \rightarrow N$  and  $\tilde{c}(0) = c(0)$ ,  $\tilde{c}(1) = c(1)$  then  $c$  and  $\tilde{c}$  define the same holonomy map up to isotopy. This shows by a standard argument that the holonomy map of  $c$  is, up to isotopy, the same holonomy map of any curve  $\tilde{c}$  homotopic to  $c$  in  $N$  with  $c(0) = \tilde{c}(0)$ ,  $c(1) = \tilde{c}(1)$ . If we now consider closed paths we obtain a map that associates to each homotopy class  $[c] \in \pi_1(N, q_o)$  (where  $q_o = c(0)$ ) the holonomy map of the path  $c: [0, 1] \rightarrow N$ . This is indeed a group homomorphism  $\text{Hol}: \pi_1(N, q_o) \rightarrow \text{Diff}^\infty(\mathbb{R}, 0)$  of the fundamental group of  $N$  based at  $q_o$  into the group of germs  $C^\infty$  diffeomorphisms fixing the origin  $0 \in \mathbb{R}$ . If we move either the base point or the discs  $\Sigma_j$  or else if we change the coverings  $\mathcal{U}$  then we obtain the same homomorphism up to conjugation in  $\text{Diff}^\infty(\mathbb{R}, 0)$ .

**Definition 2.4.** We define the *holonomy group* of the component  $N \subset$

$\text{sing}(\mathcal{F})$  as the quotient of the image of the homomorphism  $\text{Hol}: \pi_1(N, q_o) \rightarrow \text{Diff}^\infty(\mathbb{R}, 0)$  by conjugation in  $\text{Diff}^\infty(\mathbb{R}, 0)$ .

In what follows  $N \subset \text{sing}(\mathcal{F})$  is compact, connected and of Bott-Morse type. The following lemma proves the first statement in Theorem B.

**Lemma 2.5.** *If the holonomy group of the component  $N \subset \text{sing}(\mathcal{F})$  is finite then there is a neighborhood  $W$  of  $N$  in  $M$  where  $\mathcal{F}$  is given by a smooth function  $f: W \rightarrow \mathbb{R}$ .*

*Proof.* We recall (see for instance [4] Lemma 5 page 73) that a finite subgroup of  $\text{Diff}^\infty(\mathbb{R}, 0)$  is either trivial or has order two and therefore it is conjugate to the group generated by the involution  $\varphi(x) = -x$  in  $\text{Diff}(\mathbb{R}, 0)$ . Assume first that the holonomy is trivial. The proof is by a standard argument of extension by holonomy. We fix a point  $q_o \in N$  and a transverse disc  $\Sigma_{q_o}$  such that  $\mathcal{F}|_{\Sigma_{q_o}}$  is given by a Morse function  $f_o: \Sigma_{q_o} \rightarrow \mathbb{R}$  singular only at  $\{q_o\} = \Sigma_{q_o} \cap N$ . Let  $q \in N$  be given and consider a transverse disc  $\Sigma_q$  given by a transverse fibration as in the above definition of holonomy. Fix any curve  $c_q: [0, 1] \rightarrow N$  with  $c_q(0) = q_o$  and  $c_q(1) = q$ . Given a point  $y_o \in \Sigma_{q_o}$  we consider the lift  $\tilde{c}_{y_o}: [0, 1] \rightarrow L_y$  of the curve  $c$  to the leaf  $L_{y_o}$  of  $\mathcal{F}$  through the point  $y_o$ . Put  $y = \tilde{c}_{y_o}(1) \in \Sigma_q$ . We define the value  $f(y) = f_o(y_o)$ . By triviality of the holonomy of  $N$  the value  $f(y)$  does not depend on the curve  $c_q$ . Thus we can define a function  $f: W \rightarrow \mathbb{R}$  in an invariant tubular neighborhood  $W$  of  $N$  in  $M$  with the following properties:

- (i)  $f|_{\Sigma_{q_o}} = f_o$ .
- (ii)  $f$  is constant along the leaves of  $\mathcal{F}$  in  $W$ .
- (iii) The restriction  $f|_{\Sigma_q}$  to a transverse disc  $\Sigma_q$  to  $N$  at  $q \in N$  is conjugate to  $f_o$  by a holonomy map diffeomorphism  $h_{c_q}: (\Sigma_{q_o}, q_o) \rightarrow (\Sigma_q, q)$ .

And finally,

- (iv) This extension  $f$  is a smooth first integral for  $\mathcal{F}$  which is a submersion in  $W \setminus N$ .

Assume now that  $N$  has holonomy group generated by the real map  $\varphi(x) = -x$ . Then we can use the same proof of Lemma 2.5 above but replacing  $f_o$  by  $(f_o)^2$ . This function  $(f_o)^2(x) = (f_o(x))^2$  is invariant by the holonomy  $\varphi(x) = -x$  and therefore extends to a well-defined first integral for  $\mathcal{F}$  in a neighborhood  $W$  of  $N$  in  $M$ .  $\square$

**Remark 2.6.** In the case the holonomy has order 2 we cannot assure that the first integral  $f: W \rightarrow \mathbb{R}$  has connected fibers. Nevertheless, if  $\mathcal{F}$  is transversally oriented then the holonomy of  $N$  consists of orientation preserving elements in  $\text{Diff}^\infty(\mathbb{R}, 0)$  and therefore it is finite if and only if it is trivial. This shows that the order 2 case in the proof of Lemma 2.5 does not occur if  $\mathcal{F}$  is transversally oriented.

## 2.2 Proof of the Local stability

To prove Theorem B we use:

**Proposition 2.7.** *Let  $\mathcal{F}$  be a transversally orientable foliation with Bott-Morse singularities on  $M$ . Given a compact component  $N \subset \text{sing}(\mathcal{F})$  we have:*

- (i) *If  $N$  is of center type and it is a limit of compact leaves then  $N$  is stable.*
- (ii) *If  $\mathcal{F}$  is compact then  $N$  is stable of center type with trivial holonomy.*
- (iii) *If  $N$  is of center type and the holonomy group of  $N$  is finite then  $N$  is stable and the nearby leaves are all compact.*

*Proof.* First we prove (i). Suppose that  $N$  is a center and is a limit of compact leaves of  $\mathcal{F}$  say  $N = \lim_{j \rightarrow \infty} L_j$ . Fix an orientation for  $N$ . By choosing an orientation for  $\mathcal{F}$  in a tubular neighborhood of  $N$  on  $M$  we may assume that each  $L_j$  bounds a region  $R_j$  in  $M$ , this region is invariant by  $\mathcal{F}$  and therefore, because of the orientation for  $N$ , we can assume that  $N \subset R_j$  and  $R_{j+1} \subset R_j$ ,  $\forall j$  so that  $N = \lim_{j \rightarrow \infty} R_j$  as a decreasing limit: the region  $R_j$  is invariant. Since  $\lim L_j = N$  it follows that  $\lim R_j = N$  in the Hausdorff topology. Hence every neighborhood  $W$  of  $N$  in  $M$  contains  $R_j$  for  $j$  big enough; then we can take  $W' = \text{interior of } R_{j+1}$  so that  $N \subset W' \subset W$  and every leaf  $L$  of  $\mathcal{F}$  intersecting  $W'$  is contained in  $W$ . Thus  $N$  is stable.

Now we prove (ii). Assume that  $\mathcal{F}$  is compact. Then, by definition  $N$  is of center type. Since the leaves of  $\mathcal{F}$  are compact, the holonomy group  $\text{Hol}(\mathcal{F}, N) \subset \text{Diff}^\infty(\mathbb{R}, 0)$  is an orientation preserving group with finite orbits. This implies that this group is trivial and therefore  $\mathcal{F}$  has the product structure in a neighborhood of  $N$  in  $M$  (cf. Remark 2.6). Arguing as above we conclude that given any leaf  $L$  close enough to  $N$ , we may assume that each  $L$  bounds a region  $R(L)$  in  $M$ , this region

is invariant by  $\mathcal{F}$  and such that  $\lim_{L \rightarrow N} R(L) = N$ . Because of the orientation for  $N$  we can assume that the above limit is a decreasing limit so that  $N$  is stable.

Proof of (iii): As already mentioned, if  $\text{Hol}(\mathcal{F}, N)$  is finite then it is trivial and  $\mathcal{F}$  has a product structure in a neighborhood of  $N$  which implies that  $N$  is stable with compact nearby leaves.  $\square$

**Remark 2.8.** For codimension one transversally oriented *nonsingular* foliations, a *compact leaf* is stable if and only if it has trivial holonomy, this is due to Reeb [18]. Nevertheless, it is not true that a stable center type component  $N \subset \text{sing}(\mathcal{F})$  of a foliation with Bott-Morse singularities  $\mathcal{F}$  necessarily has trivial holonomy. A counterexample with a one-dimensional component  $N \subset \text{sing}(\mathcal{F})$  can be constructed as follows. Consider the sphere  $S^m$  as the gluing of  $S^1 \times D^{m-1}$  and  $D^2 \times S^{m-2}$  through the boundary. On  $S^1 \times D^{m-1}$  we consider a non-compact foliation with leaves diffeomorphic to  $\mathbb{R} \times S^{m-2}$  except for the boundary leaf diffeomorphic to  $S^1 \times S^{m-2}$  and on  $D^2 \times S^{m-2}$  we consider the trivial foliation with compact leaves  $S^1(r) \times S^{m-2}$ . These foliations glue together through the common boundary leaf  $S^1 \times S^{m-2}$ . The resulting foliation  $\mathcal{F}$  is partially depicted in Figure 2 and has a non-stable compact leaf which is diffeomorphic to  $S^1 \times S^{m-1}$ .

*Proof of Theorem B.* The first part of the theorem is exactly the content of Lemma 2.5. Assume now that the transverse type of  $\mathcal{F}$  along  $N$  is a center. Then  $N$  is stable with compact nearby leaves (cf. Proposition 2.7 (iii)). It remains to prove that these leaves are fibre bundles over  $N$  with fiber  $S^{m-n-1}$ . The local product structure and the triviality of the holonomy group give a retraction of a suitable saturated neighborhood  $W$  of  $N$  onto  $N$  having as fibers transverse discs  $\Sigma$  to  $N$ . The restriction of this retraction to any leaf  $L \subset W$  gives a proper smooth submersion of  $L$  onto  $N$ . The fibration theorem of Ehresmann [7] and the center type of  $N$  give the fibre bundle structure of  $L$ .  $\square$

### 3 Complete stability

In this section we prove the complete stability Theorem A. Our first step is the following proposition:

**Proposition 3.1.** *Let  $\mathcal{F}$  be a smooth codimension one with Bott-Morse singularities on a manifold  $M$ . Suppose that all components of  $\text{sing}(\mathcal{F})$*

are centers and there exists a compact leaf  $L_o \in \mathcal{F}$  with finite fundamental group. Then every leaf of  $\mathcal{F}$  is compact with finite fundamental group. If  $\mathcal{F}$  is transversally orientable then all leaves are diffeomorphic to  $L_o$ .

Using the 2-fold transversally orientable covering of  $\mathcal{F}$  we can assume in what follows that  $\mathcal{F}$  is transversally orientable. To prove Proposition 3.1 denote by  $\Omega(\mathcal{F})$  the union of leaves  $L \in \mathcal{F}$  which are compact with finite fundamental group and by  $\Omega(L_o)$  the connected component of  $\Omega(\mathcal{F})$  that contains the leaf  $L_o$ . Let us study the structure of  $\Omega(L_o)$ . By the Reeb local stability theorem  $\Omega(L_o)$  is open in  $M \setminus \text{sing}(\mathcal{F})$ . Since  $\Omega(L_o)$  is connected and  $\mathcal{F}$  we have that all leaves in  $\Omega(L_o)$  are diffeomorphic. We claim that  $\Omega(L_o) = M \setminus \text{sing}(\mathcal{F})$ , which obviously implies Proposition 3.1. This is an immediate consequence of the following lemma:

**Lemma 3.2.** *We have  $\partial\Omega(L_o) \subset \text{sing}(\mathcal{F})$ .*

*Proof.* We use the transverse orientability of  $\mathcal{F}$ . If there is a point  $p \in \partial\Omega(L_o) \setminus \text{sing}(\mathcal{F})$  then the leaf  $L_p \subset \partial\Omega(L_o)$  is accumulated by leaves in  $\Omega(L_o)$ , i.e., by compact leaves with finite fundamental group. We claim that  $\overline{L_p} \cap \text{sing}(\mathcal{F}) = \emptyset$ . Suppose by contradiction that there exists a component  $N \subset \text{sing}(\mathcal{F})$  such that  $N \cap \overline{L_p} \neq \emptyset$ . Then  $N \subset \overline{L_p} \setminus L_p$  and  $N$  is a center. We fix a point  $q \in N$  and a transverse disc  $\Sigma \cong \mathbb{R}^k$ , where  $k = \dim M - \dim N$ , and such that  $\Sigma \cap N = \{q\}$ . We have that  $L_p \cap \Sigma$  accumulates at  $q$ , indeed  $L_p \cap \Sigma$  defines a sequence of  $(k-1)$ -spheres  $\{S_\nu\}_{\nu \in \mathbb{N}}$ , such that in  $\Sigma$  we have  $S_{\nu+1} \geq S_\nu$  with the order given by the inclusion  $B_{\nu+1} \subset B_\nu$  and such that  $\lim_{\nu \rightarrow +\infty} S_\nu = q$ . We can assume that  $S_{\nu+1} \neq S_\nu$ . Now since  $L_p$  is accumulated by leaves in  $\Omega(L_o)$  we can similarly obtain for each fixed  $\nu \in \mathbb{N}$  a sequence of spheres  $\{S_{\nu,j}\}_{j \in \mathbb{N}}$  that satisfies  $\lim_{j \rightarrow \infty} S_{\nu,j} = S_\nu$  and such that  $S_{\nu,j} \subset L_{\nu,j}$  for some leaf  $L_{\nu,j} \subset \Omega(L_o)$ . Since the leaf  $L_{\nu,j}$  is compact we can assume that  $S_{\nu,j} = L_{\nu,j} \cap \Sigma$  and also we can assume that  $S_\nu < S_{\nu,j_\nu} < S_{\nu-1}$  for some sequence of indexes  $j_1 < j_2 < \dots < j_\nu < j_{\nu+1} < \dots$ . Put  $L_\nu := L_{\nu,j_\nu}$ .

Now, since  $S_{\nu,j_\nu} = L_\nu \cap \Sigma$  and  $L_\nu \in \Omega(L_o)$ , it follows that  $L_\nu > L_{\nu+1}$  in  $\Omega(L_o)$  and in particular every leaf  $L \in \mathcal{F}$  such that  $L \cap \Sigma$  contains a sphere  $S_{\nu,j_\nu} < S_L < S_{\nu+1,j_{\nu+1}}$  must satisfy  $L_{\nu+1} < L < L_\nu$  and in particular  $L \in \Omega(L_o)$ . In other words, the leaf  $L_p$  cannot satisfy  $L_p \cap \Sigma = \{S_\nu\}$  with  $S_{\nu+1,j_{\nu+1}} < S_\nu < S_{\nu,j_\nu}$  and  $\lim_{\nu \rightarrow \infty} S_\nu = q$ , contradiction. This shows our claim that  $\overline{L_p} \cap \text{sing}(\mathcal{F}) = \emptyset$ . This claim already shows

that  $L_p$  must be compact and since there is a finite covering  $L_\nu \rightarrow L_p$  for leaves  $L_\nu \in \Omega(L_o)$ , it follows that also  $L_p$  is compact with finite fundamental group. Thus  $\partial\Omega(L_o) \subset \text{sing}(\mathcal{F})$  completing the proof of Lemma 3.2.  $\square$

The second step in the proof of Theorem A is:

**Proposition 3.3.** *Let  $\mathcal{F}$  be a smooth codimension one foliation with Bott-Morse singularities on a closed manifold  $M^m$ ,  $m \geq 3$ . Assume that:*

- (a) *Every component  $N \subset \text{sing}(\mathcal{F})$  has center type.*
- (b) *There is a codimension  $\geq 3$  component  $N_o \subset \text{sing}(\mathcal{F})$  with finite fundamental group.*

*Then  $\mathcal{F}$  is a compact stable foliation. If  $\mathcal{F}$  is transversally oriented then its leaves are diffeomorphic.*

*Proof.* Again we can assume that  $\mathcal{F}$  is transversally orientable. By hypothesis  $\text{sing}(\mathcal{F}) \neq \emptyset$  and contains a component  $N_o^{n_o} \subset \text{sing}(\mathcal{F})$  which has a finite fundamental group and such that (b) is verified. By the Local Stability Lemma  $N_o$  is stable with compact nearby leaves. Also, since  $\mathcal{F}$  is transversally oriented, the holonomy of  $N_o$  is trivial (see Remark 2.6). This implies that each leaf of  $\mathcal{F}$  in  $W$  is diffeomorphic to a  $S^{m-n_o-1}$ -fibre bundle over  $N_o^{n_o}$ . By hypothesis (b) we have  $m-n_o \geq 3$ . Using the homotopy sequence of the fibration  $S^{m-n_o-1} \hookrightarrow L \rightarrow N_o^{n_o}$  we conclude that  $L$  has finite fundamental group. Thus the leaves of  $\mathcal{F}$  in  $W$  are compact with finite fundamental group. Since  $\mathcal{F}$  has codimension one and is transversally orientable, each such leaf has trivial holonomy and therefore (by classical Reeb Stability)  $\mathcal{F}$  has a local product structure in a neighborhood of each such leaf. In particular, these leaves are diffeomorphic to  $L$ .

Let  $\mathcal{O} = \{p \in M \setminus \text{sing}(\mathcal{F}) : L_p \text{ is diffeomorphic to } L\}$  then  $\mathcal{O} \neq \emptyset$  and  $\mathcal{O}$  is an open subset of  $M$  by the Reeb Local Stability Theorem. Indeed, the proof of the Reeb Complete Stability Theorem shows that every leaf  $L' \subset \partial\mathcal{O}$  must accumulate to a singularity of  $\mathcal{F}$ . But this is not possible because by (a) the components of  $\text{sing}(\mathcal{F})$  are stable with compact nearby leaves. Therefore  $\partial\mathcal{O} = \text{sing}(\mathcal{F})$  and  $M = \mathcal{O} \cup \text{sing}(\mathcal{F})$ . This shows that  $\mathcal{F}$  is a compact foliation.  $\square$

**Remark 3.4.** Condition (b) in Proposition 3.3 is indeed necessary. For instance, consider the foliation with Bott-Morse singularities  $\mathcal{F}$  on

$S^2 \times S^2$  given by the product of a non-periodic flow with exactly two center type singularities on  $S^2$  by the sphere  $S^2$ , which has non-compact leaves.

**Proposition 3.5.** *Let  $\mathcal{F}$  be a foliation on a closed connected manifold  $M^m$ ,  $m \geq 3$ , with Bott-Morse singularities. Assume that:*

- (i) *the transverse type of  $\mathcal{F}$  along any component  $N \subset \text{sing}(\mathcal{F})$  is a center.*
- (ii)  *$\mathcal{F}$  has some compact leaf  $L_o$  with finite fundamental group.*

*Then  $\mathcal{F}$  is a compact stable foliation.*

*Proof.* Define the set  $\Omega(\mathcal{F}) \subset M$  as the union of leaves  $L \in \mathcal{F}$  which are compact and with finite fundamental group. Then by Reeb Local Stability  $\Omega(\mathcal{F})$  is open in  $M \setminus \text{sing}(\mathcal{F})$ . Since  $L_o \in \Omega(\mathcal{F})$  it follows that  $\Omega(\mathcal{F})$  is not-empty and either  $\Omega(\mathcal{F}) = M$  (if  $\mathcal{F}$  is nonsingular) or  $\partial\Omega(\mathcal{F}) \cap \text{sing}(\mathcal{F}) \neq \emptyset$ . On the other hand, given any component  $N \subset \text{sing}(\mathcal{F})$  with  $N \cap \partial\Omega(\mathcal{F}) \neq \emptyset$  we must have  $N \subset \partial\Omega(\mathcal{F})$ . Now, since  $N \subset \partial\Omega(\mathcal{F})$  this implies that  $N$  is a limit of compact leaves of  $\mathcal{F}$  and by Theorem B the component  $N \subset \text{sing}(\mathcal{F})$  is stable. Thus all leaves close enough to  $N$  are compact with finite holonomy group and therefore there is a neighborhood  $W$  of  $N$  in  $M$  such that  $W \setminus N \subset \Omega(\mathcal{F})$  and  $\partial\Omega(\mathcal{F}) \cap W = N$ . This shows that  $\partial\Omega(\mathcal{F}) = \text{sing}(\mathcal{F})$  and therefore  $M = \Omega(\mathcal{F}) \cup \text{sing}(\mathcal{F})$ .  $\square$

The existence of the function  $f: M \rightarrow [0, 1]$  describing  $\mathcal{F}$  in Theorem A is a consequence of Proposition 3.5 to be proven below. Its proof requires the following lemma.

**Lemma 3.6.** *Let  $\mathcal{F}$  be a compact foliation on a closed manifold  $M$  having Bott-Morse singularities. Assume that  $\mathcal{F}$  is transversally orientable and  $\text{sing}(\mathcal{F}) \neq \emptyset$ . Then  $\text{sing}(\mathcal{F})$  has exactly two connected components, say  $N_1, N_2$ , and there exists an arc  $\gamma: [0, 1] \rightarrow M$  transverse to  $\mathcal{F}$  such that  $\gamma(0) \in N_1$ ,  $\gamma(1) \in N_2$ , whose image meets every leaf of  $\mathcal{F}$  at a single point.*

*Proof.* Let us first prove that  $\text{sing}(\mathcal{F})$  has at most two connected components. Take a component  $N \subset \text{sing}(\mathcal{F})$  and denote by  $\mathcal{A}(N)$  the subset of  $M$  which is the union of leaves  $L \in \mathcal{F}$  such that  $L$  bounds a region  $R(L) \subset M$  containing  $N$  and such that  $\mathcal{F}|_{R(L) \setminus L}$  is a fibre bundle over  $N$ . Clearly  $N \subset \partial\mathcal{A}(N)$  and  $\partial R(L) \subset \text{sing}(\mathcal{F})$ . Suppose that there is a component  $N' \subset \partial\mathcal{A}(N) \setminus N$ ; let us prove that

$M = N \cup \mathcal{A}(N) \cup N'$ . First we observe that  $S = N \cup \mathcal{A}(N) \cup N'$  is an open subset of  $M$ . To see this take an invariant neighborhood  $W$  of  $N'$  given by the local stability theorem for  $N'$ . Since  $N' \subset \partial\mathcal{A}(N)$  there is a leaf  $L \subset \mathcal{A}(N)$  which intersects  $W$  and therefore is entirely contained in  $W$ . By definition of  $\mathcal{A}(N)$  the leaf  $L$  bounds a region  $R(L) \subset M$  such that  $\mathcal{F}|_{R(L) \setminus N}$  is a fiber bundle over  $N$  and by the choice of  $W$ ,  $L$  bounds a region  $R'(L) \subset W$  such that  $\mathcal{F}|_{R'(L) \setminus N'}$  is a fiber bundle over  $N'$ . Finally, since  $\mathcal{F}$  has a local product structure in a neighborhood of  $L$  we conclude that  $W \setminus N' \subset \mathcal{A}(N)$ . Thus  $S$  is open in  $M$ . The above arguments also show that  $S$  contains the union of two compact submanifolds with boundary which are glued along their common boundary ( $L$  above). Hence  $S$  equals  $M$  and therefore  $\text{sing}(\mathcal{F})$  cannot have more components.

To construct the arc  $\gamma$  in the statement we first need:

**Claim 1.** *Let  $\gamma_o: S^1 \rightarrow M$  be a closed curve transverse to  $\mathcal{F}$  and to  $\text{sing}(\mathcal{F})$ . Then  $\gamma_o$  intersects all leaves of  $\mathcal{F}$  and all components of  $\text{sing}(\mathcal{F})$ .*

*Proof of Claim 1.* Denote by  $\Omega$  the set of all leaves  $L \in \mathcal{F}$  such that  $\gamma_o \cap L \neq \emptyset$ . By transversality this is an open set. To see this set is also closed in  $M \setminus \text{sing}(\mathcal{F})$  take a nonsingular point  $p \in \partial\Omega$  and choose an invariant neighborhood  $W$  of the leaf  $L_p$  given by the local stability where  $\mathcal{F}$  is trivial. In  $W$  any transverse curve to  $\mathcal{F}$  intersects all leaves. Since  $p \in \partial\Omega$  we have  $\gamma_o \cap W \neq \emptyset$  and therefore  $\gamma_o \cap L_p \neq \emptyset$ . Thus  $\Omega$  is closed in  $M \setminus \text{sing}(\mathcal{F})$  and  $\Omega = M \setminus \text{sing}(\mathcal{F})$ . Similar arguments prove that  $\gamma_o$  intersects each component of  $\text{sing}(\mathcal{F})$ .  $\square$

Let  $X$  be a vector field transverse to  $\mathcal{F}$  on  $M$ . Let  $N \subset \text{sing}(\mathcal{F})$  be given, we can assume that  $X$  is radial pointing outwards in a neighborhood of  $N$ . Consider a point  $p \in N \subset \text{sing}(\mathcal{F})$  and the orbit  $\gamma$  of  $X$  whose  $\alpha$ -limit is  $p$ . We consider the  $\omega$ -limit  $\omega(\gamma)$ . Then  $\omega(\gamma)$  avoids a neighborhood of  $N$ . In fact we have

**Claim 2.**  $\omega(\gamma) = \{q\}$  where  $q \in \text{sing}(\mathcal{F}) \setminus N$ .

*Proof of Claim 2.* Suppose  $\omega(\gamma)$  contains some non-singular point  $q$ . Then  $\gamma$  cuts the leaf  $L_q$  infinitely many times. Let us choose two such points  $p_1 = \gamma(t_1)$  and  $p_2 = \gamma(t_2)$ ,  $t_2 > t_1$  close enough to  $q$  so that they avoid a neighborhood of  $N$  and a path  $\beta: [0, 1] \rightarrow L_q$  joining  $p_1$  to  $p_2$ .



By a classical argument  $\exists \delta > 0$  and a smooth closed curve  $\gamma_o$ , transverse to  $\mathcal{F}$  such that  $\gamma$  contains the arc  $C = \alpha([t_1 + \delta, t_2 - \delta])$  and the complement  $\gamma_o([0, 1]) \setminus C$  projects onto  $\beta$  via a transverse fibration with basis  $\beta$ . Thus we can construct a closed curve  $\gamma_o$  transverse to  $\mathcal{F}$  and which avoids a neighborhood of  $N$ . This contradicts Claim 1. Therefore  $\omega(\gamma) \subset \text{sing}(\mathcal{F})$ . Since  $X$  is radial in a neighborhood of  $\text{sing}(\mathcal{F})$  we have that  $\omega(\gamma)$  has to be a single point, say  $\omega(\gamma) = q \in \text{sing}(\mathcal{F})$ . Because  $X$  points outwards in a neighborhood of  $N$  we have that  $q$  cannot belong to  $N$ , which proves the claim.  $\square$

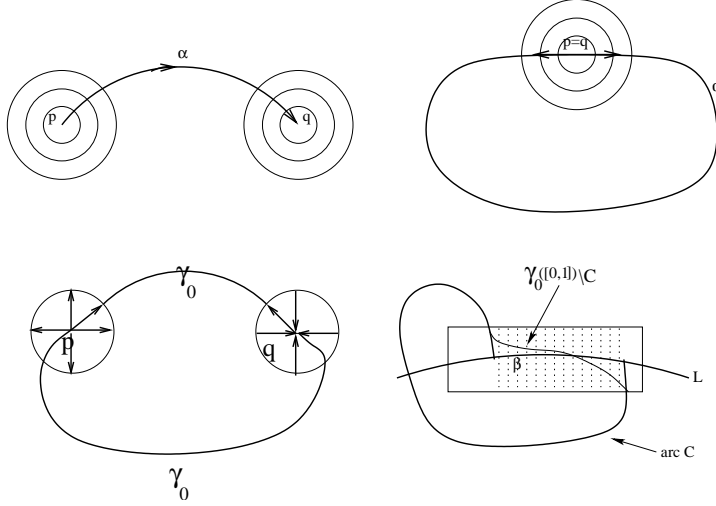


Figure 1:

Notice that this implies, in particular, that  $\text{sing}(\mathcal{F})$  has at least (and therefore exactly) two components  $N_1, N_2$ . By Claim 2 we have an arc  $\gamma_o: [0, 1]: M$  such that  $\gamma_o(0) \in N_1$  and  $\gamma_o(1) \in N_2$  and  $\gamma(0, 1)$  is everywhere transverse to  $\mathcal{F}$ . As in the proof of Claim 1 one can prove that  $\gamma_o$  intersects each leaf of  $\mathcal{F}$ . Also by compactness and transversality  $\gamma_o$  intersects each leaf of  $\mathcal{F}$  a finite number of times.

If  $\gamma_o$  cuts a fixed leaf  $L_o$  of  $\mathcal{F}$  a certain number  $k \geq 2$  of times then the proof of Claim 2 above shows how to modify  $\gamma_o$  into a curve  $\gamma'_o$  that cuts  $L_o$  only  $k - 1$  times. Therefore we may assume that  $\gamma_o$  cuts  $L_o$  exactly one time.

**Claim 3.**  $\#(\gamma_o \cap L) = 1$ , for each leaf  $L$  of  $\mathcal{F}$ .

*Proof of Claim 3.* Let  $\mathcal{O}$  be the set of points  $x \in M \setminus \text{sing}(\mathcal{F})$  such that  $\#(\gamma_o \cap L_x) = 1$ . By Local Stability we have a local product structure for  $\mathcal{F}$  around any compact leaf  $L$  and therefore  $\mathcal{O}$  is open in  $M \setminus \text{sing}(\mathcal{F})$ . We claim that  $\partial\mathcal{O} = \text{sing}(\mathcal{F})$ . Assume by contradiction that there is a leaf  $L \subset \partial\mathcal{O}$ . Then again by the local product structure we have  $\#(\gamma_o \cap L_x) = \#(\gamma_o \cap L)$ ,  $\forall x$  close enough to  $L$  and therefore we get a contradiction. This shows that  $M = \mathcal{O} \cup \text{sing}(\mathcal{F})$  and proves the claim.  $\square$

Now, by the local structure of  $\mathcal{F}$  around the singularities we obtain that also  $\#(\gamma_o \cap N) = 1$  for any component  $N \subset \text{sing}(\mathcal{F})$ . This ends the proof of the lemma.  $\square$

Theorem A is now an immediate consequence of Propositions 3.3 and 3.5 and Lemma 3.6.

### 3.1 Examples and remarks on stability

The condition on the codimension of the singular set in Theorem A is necessary, for otherwise one can construct examples of foliations with Bott-Morse singularities with only center type singularities and non-compact leaves, as for instance the example in Remark 2.8. A modification of that construction yields to a foliation with center type singularities which are accumulated by compact leaves and also by non-compact leaves. For this, let  $A^m$  be a compact annulus (*i.e.*, an  $m$ -disc minus a smaller  $m$ -disc in its interior), and consider a foliation  $\mathcal{F}_A$  in  $S^1 \times A^m$  tangent to the boundary  $\partial(S^1 \times A^m) = (S^1 \times S_1^{m-1}) \cup (S^1 \times S_2^{m-1})$ , transverse to the annuli  $\{z\} \times A^m$ ,  $z \in S^1$  and such that each restriction  $\mathcal{F}_A|_{\{z\} \times A^m}$  is equivalent to the trivial foliation by  $(m-1)$ -spheres concentric and tangent to the boundary of  $A^m$ . We may also choose  $\mathcal{F}_A$  so that each leaf on  $S^1 \times A^m$ , outside the boundary, is non-compact and accumulates both components of  $\partial(S^1 \times A^m)$  as  $\mathcal{F}_o$  above. Now we consider a sequence of positive numbers  $1 = r_1 > r_2 > \dots > r_j > r_{j+1} > \dots$  converging to zero. Let  $A_j$  be the annulus of internal radius  $r_{j+1}$  and external radius  $r_j$ . On each solid annulus we put a copy  $\mathcal{F}_{A_j}$  of  $\mathcal{F}_A$ . Glue all these foliations in a foliation  $\mathcal{F}'_o$  of the product  $S^1 \times D^m$  to get a foliation there, with singular set  $S^1 \times \{pt\}$  of center type. Finally glue two copies of  $\mathcal{F}'_o$  into a foliation  $\mathcal{F}$  of  $S^1 \times S^m$  with two circles  $N_1, N_2$  as singular set, both with center types. Each component  $N_j$  is accumulated by compact leaves (diffeomorphic to  $S^1 \times S^{m-2}$ ) and also by noncompact leaves (diffeomorphic to  $\mathbb{R} \times S^{m-2}$ ) as well. In particular,

$N_j$  is *not* stable and  $\mathcal{F}$  is not compact, although  $N_j$  is of center type and accumulated by compact leaves.

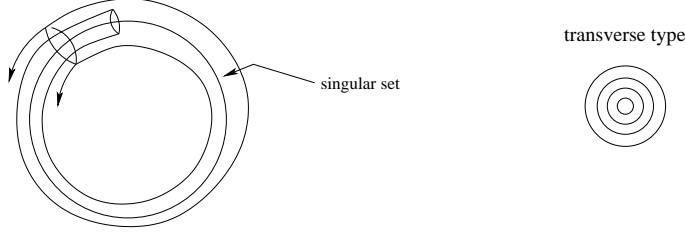


Figure 2:

**Example 5.** We decompose  $S^3$  as the union of two solid torii  $S^3 = (S_1^1 \times D_1^2) \cup (D_2^2 \times S_2^1)$  with common boundary  $S_1^1 \times S_2^1$ . In the solid torus  $S_1^1 \times D_1^2$  we consider the product foliation  $S^1 \times \mathcal{C}$  where  $\mathcal{C}$  is the foliation of the 2-disc by concentric circles. We decompose the second solid torus as the union of a solid torus and a solid annulus with common boundary  $S_3^1 \times S_2^1$ , i.e.,  $D_2^2 \times S_2^1 = (A^2 \times S_2^1) \cup (D_3^2 \times S_2^1)$ . In the solid torus  $D_3^2 \times S_2^1$  we put another trivial foliation  $\mathcal{C} \times S^1$ . Finally, in the solid annulus  $A^2 \times S_2^1$  we consider a product foliation  $\mathcal{G} \times S_2^1$  where  $\mathcal{G}$  is a one-dimensional foliation in  $A^2$  as follows:

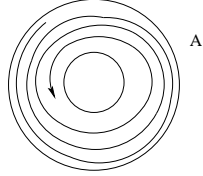


Figure 3:

$\mathcal{G}$  has a noncompact leaf accumulating the two circles in the boundary of  $A^2$ . Gluing all together we obtain a foliation  $\mathcal{F}$  on  $S^3$  with singular set  $\text{sing}(\mathcal{F}) = \text{union of two circles which are stable with respect to } \mathcal{F}$ , however  $\mathcal{F}$  is not a foliation by compact leaves due to the noncompact leaves in  $A^2 \times S_2^1$ . This construction cannot be performed for dimension  $m \geq 4$  as it is implied by Proposition 3.5.

Let now  $\mathcal{F}$  and  $f$  be as in Theorem A and assume  $m \geq 4$ . If  $\text{sing}(\mathcal{F})$  has some isolated singularity then by Reeb complete stability theorem

all leaves are diffeomorphic to  $S^{m-1}$ . Nevertheless,  $\text{sing}(\mathcal{F})$  is not necessarily of dimension zero. For instance, take the classical Hopf fibration of  $S^3$  over  $S^2$  with fiber  $S^1$ . Now consider the corresponding disc bundle over  $S^2$ . Its total space  $E^4$  is a four dimensional manifold with boundary  $S^3$ . Using the discs  $D^2$  bounded by the fibers we can construct a foliation with Bott-Morse singularities  $\mathcal{F}_1$  of  $E^4$  having compact leaves diffeomorphic to  $S^3$  and singular set  $S^2 \cong \mathbb{C}P(1)$ . Now glue to  $E^4$  a four dimensional disc in the obvious way to obtain the complex projective plane  $\mathbb{C}P(2)$  and a foliation with Bott-Morse singularities  $\mathcal{F}$  of  $\mathbb{C}P(2)$  with leaves  $S^3$  and singular set  $S^2$  union a point. This same construction generalizes to  $\mathbb{C}P(n)$  regarded as the union of a  $2n$ -disc and  $\mathbb{C}P(n-1)$ .

## 4 Compact foliations with Bott-Morse singularities

Let  $\mathcal{F}$  be a transversally oriented, compact foliation with Bott-Morse singularities on the closed, oriented, connected manifold  $M^m$ ,  $m \geq 3$ . Notice that Proposition 2 implies that each leaf  $L$  of  $\mathcal{F}$  and each component  $N$  of the singular set is stable with finite holonomy. By Lemma 3.6 one has Theorem C as an immediate consequence. This obviously imposes stringent conditions on both, the topology of  $M$  and  $L$ . Let us see what this says when  $M$  has dimensions 3 and 4. If  $m = 3$ , then  $L$  must be a two-dimensional closed oriented manifold that fibers over another manifold of dimension 0 or 1, with fiber a sphere. The only possibilities for  $L$  are to be  $S^2$ , fibered over a point, or the 2-torus  $T = S^1 \times S^1$ , since there are no other  $S^1$ -bundles over  $S^1$ , except for the Klein bottle which is not orientable. Hence the possibilities for the double-fibration in Theorem C are:

- (i) If  $N_1$  is a point, then  $L$  must be a 2-sphere  $S^2$ , and this surface does not fiber over  $S^1$ , hence  $N_2$  must be also a point. This is the classical case envisaged by Reeb and others, the leaves are copies of  $S^2$  and  $M$  is the 3-sphere, regarded as the suspension over  $S^2$ .
- (ii) If  $N_1$  is a circle, then  $L$  is the torus  $T = S^1 \times S^1$  and  $M$  is the result of gluing together two solid torii along their common boundary. The manifolds one gets in this way are either orientable  $S^1$ -bundles over  $S^2$  (and there is one such bundle for each integer, being classified by their Euler class), or a lens space  $L(p, q)$ , obtained by

identifying two solid torii by a diffeomorphism of their boundaries that carries a meridian into a curve of type  $(p, q)$  in  $T$ .

We remark that the hypothesis of having a compact foliation is necessary, otherwise the Theorem D does not hold. For instance, decompose  $S^3$  as a union of two solid torii  $T_1, T_2$ , as usual. Foliate  $T_1 = S^1 \times D^2$  by concentric torii  $S^1 \times S^1$ , and put Reeb's foliation on  $T_2$ . We get a foliation on  $S^3$  with singular set a circle of center type.

Notice that Theorem C implies:

**Theorem D.** *Let  $M$  be a closed oriented connected 3-manifold equipped with a transversely oriented compact foliation  $\mathcal{F}$  with Bott-Morse singularities. Then either  $\text{sing}(\mathcal{F})$  consists of two points, the leaves are 2-spheres and  $M$  is  $S^3$ , or  $\text{sing}(\mathcal{F})$  consists of two circles, the leaves are torii and  $M$  is homeomorphic to a Lens space or to an  $S^1$ -bundle over  $S^2$ .*

Examples 1 and 2 show that all  $S^1$ -bundles over  $S^2$  and all Lens spaces admit compact foliations as in Theorem D.

When  $m = 4$  the list of possibilities for  $L$  and  $M$  is larger. For instance, we can foliate  $S^4$  in various ways:

- By 3-spheres with two isolated centers.
- By copies of  $S^1 \times S^2$  with two circles as singular set.
- Think of  $S^4$  as being the space of real  $3 \times 3$  symmetric matrices  $A$  of trace zero and  $\text{tr}(A^2) = 1$ . The group  $SO(3, \mathbb{R})$  acts on  $S^4$  by  $A \mapsto O^t A O$ , for a given  $O \in SO(3, \mathbb{R})$  and  $A \in S^4$ . As noticed in [9] this gives an isometric action of  $SO(3, \mathbb{R})$  on the sphere  $S^4$  with two copies of  $\mathbb{R}P(2)$  as singular set. The leaves are copies of the flag manifold

$$F^3(2, 1) \cong SO(3, \mathbb{R})/(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong L(4, 1)/(\mathbb{Z}/2\mathbb{Z}),$$

of (unoriented) planes in  $\mathbb{R}^3$  and lines in these planes.

- Now consider the complex projective plane  $\mathbb{C}P(2)$ . Thinking of it as being  $\mathbb{C}^2$  union the line at infinity, one gets a foliation by copies of  $S^3$  with an isolated singularity at the origin and a copy of  $S^2 \cong \mathbb{C}P(1)$  at infinity.

- Notice that, as in Example 3, the group  $SO(3, \mathbb{R})$  is a subgroup of  $SO(3, \mathbb{C})$  and therefore acts on  $\mathbb{C}P(2)$  in the usual way. The orbits of this action are copies of the Flag manifold  $F_+^3(2, 1) \cong SO(3, \mathbb{R})/(\mathbb{Z}/2\mathbb{Z})$ , which is a double cover of  $F^3(2, 1)$ . The singular set now consists of the quadric  $\sum_{j=0}^2 z_j^2 = 0$ , which is diffeomorphic to  $S^2$ , and a copy of  $\mathbb{R}P(2)$ . As in Example 3, this foliation is mapped to the above foliation of  $S^4$  by the projection  $\mathbb{C}P(2) \rightarrow \mathbb{C}P(2)/j \cong S^4$ , where  $j: \mathbb{C}P(2) \rightarrow \mathbb{C}P(2)$  is complex conjugation (by [2], [1] or [11]).

Let us discuss the various possibilities for  $L$  and  $M$ . Let  $N_1$  and  $N_2$  be the connected components of  $\text{sing}(\mathcal{F})$ . If  $N_1$  is a point then each leaf  $L$  must be  $S^3$ . We claim that there are three possibilities for  $N_2$ : it can be either a point, the 2-sphere or the projective plane  $\mathbb{R}P(2)$ . Indeed,  $L$  fibers over  $N_2$  with fiber a sphere, and  $S^3$  does not fiber over  $S^1$ . This implies that  $N_2$  has cannot have dimension one. If  $N_2$  has dimension two then necessarily is diffeomorphic to  $S^2$  or to  $\mathbb{R}P(2)$ . Thus the possibilities are the following:

- (i.a) If  $N_2$  is also a point, then  $M$  is  $S^4$  by Reeb's theorem.
- (i.b) If  $N_2$  is the 2-sphere then one has a fiber bundle

$$S^1 \hookrightarrow S^3 \longrightarrow N_2;$$

such a bundle necessarily corresponds to a free  $S^1$ -action on  $S^3$ . The effective actions of  $S^1$  on 3-manifolds are classified in [14], and the only free action on  $S^3$  is the usual one, which yields to the Hopf fibration  $S^1 \hookrightarrow S^3 \longrightarrow S^2$ , and  $M$  is the complex projective plane  $\mathbb{C}P^2$ . Of course the projection  $S^2 \rightarrow \mathbb{R}P(2)$  yields to a fibre bundle  $S^1 \hookrightarrow S^3 \longrightarrow \mathbb{R}P(2)$ .

- (ii) If  $N_1$  is a circle, then  $L$  fibers over  $S^1 \cong N_1$  with fiber a 2-sphere, so  $L$  is  $S^1 \times S^2$ , and  $N_2$  can be either a circle  $S^1$ ,  $S^2$  or  $\mathbb{R}P(2)$ . If  $N_2 \cong S^1$  then both fibrations  $L \xrightarrow{\pi_i} N_i$ ,  $i = 1, 2$ , necessarily coincide. Then  $M$  is the result of taking two copies of the corresponding disc bundle, and glued them along their common boundary  $L$  by some diffeomorphism. If  $N_2$  is  $S^2$  or  $\mathbb{R}P(2)$  then  $L$  is a product  $S^1 \times S^2$ .
- (iii) If  $N_1$  and  $N_2$  are both surfaces, then they can be oriented or not, and  $L$  is a closed, oriented Seifert manifold. The manifolds  $N_1$  and  $N_2$  can not be arbitrary, since  $L$  must fiber over both of them

simultaneously, but there is a lot of freedom. For instance, notice that we can use the procedure in Example 2 to construct compact foliations with Bott-Morse singularities whenever we have a double-fibration as in Theorem C, regardless of whether or not the hypothesis of Theorem A are satisfied.

## References

- [1] M. F. Atiyah, J. Berndt: *Projective varieties, Severi varieties and spheres*; Surveys in Differential Geometry, Vol. VIII, Boston, MA, 2002, 1–27.
- [2] M. F. Atiyah, E. Witten: *M-theory dynamics on a manifold of  $G_2$  holonomy*; Adv. Theor. Math. Phys. 6 (2002), no 1, 1–106.
- [3] R. Bott: *Nondegenerate critical manifolds*; Annals of Math. vol. 60, # 2, 1954, p. 248–261.
- [4] C. Camacho, A. Lins Neto: *Geometric theory of foliations*. Translated from the Portuguese by Sue E. Goodman. Birkhäuser Boston, Inc., Boston, MA, 1985.
- [5] C. Camacho, B. Scárdua: *On codimension one foliations with Morse singularities*; to appear.
- [6] J. P. Dufour: *Linéarisation de certaines structures de Poisson*; J. Diff. Geom. 32 (1990), 415–428.
- [7] C. Godbillon: *Feuilletages, Études géométriques*, Birkhäuser.
- [8] D. Haibao, E. Rees: *Functions whose critical set consists of two connected manifolds*; Bol. Soc. Mat. Mex., vol. 37, 1992, p. 139–149.
- [9] W. Y. Hsiang, B. H. Lawson: *Minimal submanifolds of low cohomogeneity*; J. Differential Geometry 5 (1971), 1–38.
- [10] A. Kollross: *A classification of hyperpolar and cohomogeneity one actions*; Trans. Amer. Math. 354 (2001), 571–612.
- [11] D. T. Lê, J. Seade, A. Verjovsky: *Quadric, orthogonal actions and involutions in complex projective spaces*; Ens. Math. 49 (2003), no. 1–2, 173–203.
- [12] J. Milnor: *Morse Theory*; Ann. of Math. Studies, Princeton 1968.
- [13] M. Morse: *The calculus of variations in the large*; American Math. Soc. Colloquium Publications, 18. AMS, Providence, RI, 1996.

- [14] F. Raymond: *Classification of the actions of the circle on 3-manifolds*; Trans. Amer. Math. Soc. **131**, 1968, 51–78.
- [15] G. Reeb: *Variétés feuilletées, feuiller voisines*; C.R.A.S. Paris 224, 1947, p. 1613-1614.
- [16] G. Reeb: *Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique*; C.R.A.S. Paris 222, 1946, p. 847-849.
- [17] G. Reeb: *Sur certaines propriétés topologiques des variétés feuilletées*; (Thesis), Publ. Inst. Math. Univ. Strasbourg 11, p. 5-85, 155-156.
- [18] G. Reeb: *Sur certaines propriétés topologiques des variétés feuilletées*; Publ. Inst. Math. Univ. Strasbourg 11, pp. 5–89, 155–156. Actualités Sci. Ind., no. 1183 Hermann & Cie., Paris, 1952.
- [19] B. Scárdua, J. Seade: *Codimension one foliations with Bott-Morse singularities II*; in preparation.
- [20] H. J. Sussmann: *Orbits of families of vector fields and integrability of distributions*; Trans. Amer. Math. Soc. 180 (1973), 171-188.
- [21] A. Weinstein: *The local structure of Poisson manifolds*; J. Diff. Geom. 18 (1983), 523-557.

Bruno Scárdua  
 Instituto de Matemática  
 Universidade Federal do Rio de Janeiro  
 Caixa Postal 68530  
 Cidade Universitária  
 21.945-970 Rio de Janeiro-RJ  
 BRAZIL  
  
 scardua@impa.br

José Seade  
 Instituto de Matemáticas  
 UNAM  
 Unidad Cuernavaca  
 Av. Universidad s/n,  
 Colonia Lomas de Chamilpa  
 C.P. 62210, Cuernavaca, Morelos  
 MÉXICO  
  
 jseade@matcuer.unam.mx